On solving a class of unbalanced Ermakov-Pinney systems

2001 J. Phys. A: Math. Gen. 34 L563

(http://iopscience.iop.org/0305-4470/34/42/101)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) L563–L566 PII: S0305-4470(01)26305-7

LETTER TO THE EDITOR

On solving a class of unbalanced Ermakov–Pinney systems

Chris Athorne

Department of Mathematics, University of Glasgow, Glasgow G12 8QW, UK

Received 26 June 2001, in final form 16 August 2001
Published 12 October 2001
Online at stacks.iop.org/JPhysA/34/L563

Abstract
We use an intertwining property of linear differential operators to construct the general solution of a type of coupled Ermakov–Pinney system with two distinct frequency terms.

PACS numbers: 02.30.Ik, 02.30.Uu, 05.45.Yv

1. Introduction

By an *unbalanced Ermakov system* we mean a coupled fourth-order system of the form

$$\begin{aligned} \frac{d^2x}{dt^2} + \omega_1^2 x &= x^{-3} f(x/y) \\ \frac{d^2y}{dt^2} + \omega_2^2 y &= y^{-3} g(y/x) \end{aligned} \tag{1}$$

where f and g are arbitrary functions of their arguments and where $\omega_1 \neq \omega_2$. When $\omega_1 = \omega_2$ we will call the system *balanced*. Systems of the second type are much discussed in the literature and their modern treatment goes back to [8].

A crucial property of balanced Ermakov systems with $\omega_1 = \omega_2 = \omega(t)$, a function of the independent variable t alone, is that they possess an invariant and are susceptible to a hodograph-type transformation which linearizes them [1] so that they are integrable in terms of solutions to $\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$. A great deal of information about their solutions can be gleaned in this way [2, 3]. They are effectively linear extensions of Hamiltonian systems on the sphere [4]. Systems with a more general form of ω depending on, in addition to t, the dependent variables and their derivatives have been the subject of much study. In particular the authors of [5] have extended the linearizable class in this direction.

Such methods do not apply to the unbalanced systems, chiefly because the invariant is no longer available and we should emphasise that from one point of view, therefore, they are not Ermakov systems at all.

Ermakov–Pinney systems are those for which the right-hand sides of equations (1) are finite sums of terms of the form $x^i y^j$ where $i + j + 3 = 0$. This homogeneity of weight -3 is a crucial property of general Ermakov systems.

It would be nice to be able to treat the unbalanced systems with the same algebraic elegance as the symmetric but this really looks to be too much to ask in most cases. Nevertheless, as we will show below, there is at least one restricted class, namely,

$$\begin{aligned}\frac{d^2x}{dt^2} + \omega_1^2(t)x &= \lambda x^{-1}y^{-2}\\ \frac{d^2y}{dt^2} + \omega_2^2(t)y &= \lambda x^{-2}y^{-1}\end{aligned} \tag{2}$$

which can be solved in terms of the solutions of $\frac{d^2\theta}{dt^2} + \omega_1^2(t)\theta = 0$ and $\frac{d^2\tilde{\theta}}{dt^2} + \omega_2^2(t)\tilde{\theta} = 0$.

The construction hinges on the observation that system (2) is equivalent to the intertwining condition

$$(\partial^2 + \omega_2^2)(\alpha\partial + \beta) = (\alpha\partial + \gamma)(\partial^2 + \omega_1^2). \tag{3}$$

The operator $\alpha\partial + \beta$ then maps $\ker(\partial^2 + \omega_1^2)$ to $\ker(\partial^2 + \omega_2^2)$. Since the functions α and β here are constructed from x and y in (2) we have relations which can be solved for x and y in terms of bases of the kernels. This allows us to reduce the solution of the unbalanced Ermakov–Pinney equation to a single quadrature.

In the next section we present the details of the general calculation and then some remarks.

2. Intertwining

The intertwining relation (3) in the case that $\omega_1 = \omega_2 = \omega$ expresses the condition that $L = \alpha\partial + \beta$ represents a Lie point symmetry of the linear operator $\partial^2 + \omega^2$. Some remarks on this fact and on its connection with differential Galois theory are to be found in [7]. In this case it is found that $L = \phi^2\partial - \phi\dot{\phi}$ where ϕ solves the Ermakov–Pinney equation

$$\frac{d^2\phi}{dt^2} + \omega^2\phi = k\phi^{-3}. \tag{4}$$

So solutions of (4) correspond to point symmetries of the linear operator $\partial^2 + \omega^2$. But conversely, we may solve (4) using the fact that L maps the kernel of $\partial^2+\omega^2$ onto itself. If θ_1 and θ_2 are linearly independent elements of $\ker(\partial^2 + \omega^2)$, with unit Wronskian $\frac{d\theta_2}{dt}\theta_1 - \theta_2\frac{d\theta_1}{dt} = 1$, there exist, for each ϕ, constants c_1 and c_2 such that

$$\phi^2\frac{d\theta_1}{dt} - \phi\frac{d\phi}{dt}\theta_1 = c_1\theta_1 + c_2\theta_2. \tag{5}$$

We solve this relation for ϕ using θ_1^{-3} as integrating factor and noting that $\theta_1^{-2} = \frac{d}{dt}\left(\frac{\theta_2}{\theta_1}\right)$ to yield

$$\phi^2 = d_1\theta_1^2 + 2d_2\theta_1\theta_2 + d_3\theta_2^2 \tag{6}$$

for constants $d_2 = -c_1$, $d_3 = -c_2$ and d_1 a constant of integration. By substitution one verifies that $d_1d_2 - d_2^2 = k$ and one has the familiar general solution to (4).

The unbalanced case (3) requires $\gamma = \beta + 2\frac{d\alpha}{dt}$ and, after a single integration, relations of the following form between α and β:

$$\frac{d^2\alpha}{dt^2} + (\omega_2^2 - \omega_1^2)\alpha + 2\frac{d\beta}{dt} = 0 \tag{7}$$

$$\beta\frac{d\alpha}{dt} - \alpha\frac{d\beta}{dt} + \beta^2 + \alpha^2\omega_1^2 = k \tag{8}$$

for constant k. These may be written in a more symmetric form by putting $\beta = \alpha\phi$ and $\tilde{\phi} = -\phi - \alpha^{-1}\frac{d\alpha}{dt}$:

$$\omega_1^2 = \frac{k}{\alpha^2} + \frac{d\phi}{dt} - \phi^2 \tag{9}$$

$$\omega_2^2 = \frac{k}{\alpha^2} + \frac{d\tilde{\phi}}{dt} - \tilde{\phi}^2. \tag{10}$$

Finally, introducing x and y by $\phi = -x^{-1}\frac{dx}{dt}$ and $\tilde{\phi} = -y^{-1}\frac{dy}{dt}$ does the job and we recover the system (2), noting that α is xy up to a multiplicative constant. The choice $\alpha = xy$ leads to the identification $\lambda = k$.

The operator $L = xy\partial - y\frac{dx}{dt}$ maps a basis $\{\theta_1, \theta_2\}$ of $\ker(\partial^2 + \omega_1^2)$ into a basis $\{\tilde{\theta}_1, \tilde{\theta}_2\}$ of $\ker(\partial^2 + \omega_2^2)$:

$$\begin{aligned} y\left(x\frac{d\theta_1}{dt} - \theta_1\frac{dx}{dt}\right) &= \tilde{\theta}_1 \\ y\left(x\frac{d\theta_2}{dt} - \theta_2\frac{dx}{dt}\right) &= \tilde{\theta}_2 \end{aligned} \tag{11}$$

and solving this system for x and y yields

$$x^{-1}\frac{dx}{dt} = \begin{vmatrix} \theta_1 & \theta_2 \\ \tilde{\theta}_1 & \tilde{\theta}_2 \end{vmatrix}^{-1} \begin{vmatrix} d\theta_1/dt & d\theta_2/dt \\ \tilde{\theta}_1 & \tilde{\theta}_2 \end{vmatrix} \tag{12}$$

$$xy = \begin{vmatrix} \theta_1 & \theta_2 \\ \tilde{\theta}_1 & \tilde{\theta}_2 \end{vmatrix}. \tag{13}$$

If we take $\{\theta_1, \theta_2\}$ to be any fixed basis of $\ker(\partial^2 + \omega_1^2)$, then $\{\tilde{\theta}_1, \tilde{\theta}_2\}$ can be one member of the four-parameter family of bases of $\ker(\partial^2 + \omega_2^2)$. These four constants, together with the single constant of integration arising from (12), will satisfy a single algebraic relation involving λ and so x and y will depend on the requisite four constants of integration and constitute the general solution of (2).

This relation can be made explicit. Note firstly that the scaling transformation $x \mapsto \mu x$, $y \mapsto \mu^{-1}y$ preserves the system (2). This μ corresponds to the constant of integration arising from equation (12). Now let the basis $\{\theta_1, \theta_2\}$ have unit Wronskian. Because relations (11) are unaltered by the scaling transformation, the four parameters in the family of bases $\{\tilde{\theta}_1, \tilde{\theta}_2\}$ are independent of μ. But differentiating (11) and forming the Wronskian of the basis $\{\tilde{\theta}_1, \tilde{\theta}_2\}$ shows that this Wronskian is equal to λ. This is the algebraic relation in question.

3. Remarks

The use of the intertwining relation has reduced the problem (2) to a single quadrature up to the description of the kernels of the linear operators which is, of course, not explicitly tractable in general. Even where it is tractable the quadrature of the first order, homogeneous, linear equation (12), depending as it does on three constant parameters, is unlikely to be so. Nevertheless one can extract information about the asymptotics of solutions at singularities.

The intertwining relation (3) is a generalization of the Darboux map [6] used in soliton theory. It is also a *deformation* of the relation obtaining when $\omega_1 = \omega_2$ where it corresponds to a Lie point symmetry condition. The Ermakov–Pinney equation and system discussed above are therefore intimately connected with the symmetries of the family of linear, second-order operators: there is no restriction, beyond regularity, on the t dependence of the 'frequency' functions.

But it is natural to ask whether some more generality can be achieved by allowing (3) to be a relation between 2×2 matrix differential operators:

$$(I\partial + H)(A\partial + B) = (A\partial + C)(I\partial + K). \tag{14}$$

Here I is the unit 2×2 matrix and H, A, B, C and K general 2×2 matrices. It is straightforward to show that the $A\partial$ in the intertwining operators is immaterial and we may, without loss of generality, consider

$$(I\partial + H)\Theta = \Theta(I\partial + K). \tag{15}$$

One may further show that the matrix Θ equals $\Theta_1\Theta_2^{-1}$ where $(\partial+H)\Theta_1 = (\partial+K)\Theta_2 = 0$ so that Θ_i and Θ_2 are arbitrary fundamental solution matrices of these linear systems.

The form of H and K in (15), however, is still subject to gauge transformations

$$\begin{aligned}\partial + H &\rightarrow g^{-1}(\partial + H)g \\ \partial + K &\rightarrow k^{-1}(\partial + K)k\end{aligned}$$

g and k being invertible 2×2 matrices whose entries are independent of the solution matrices Θ_1 and Θ_2. By using diagonal g and k involving exponentials of integrals of the diagonal entries of H and K we may reduce H and K to off-diagonal form. Then, by using g and k of the form

$$\begin{pmatrix} a & \mathrm{d}a/\mathrm{d}t \\ 0 & a^{-1} \end{pmatrix}$$

for suitable a in each case, we may take the lower corner entries in H and K to be constant. We are then left with the system forms of the operators $\partial^2 + \omega_1^2$ and $\partial^2 + \omega_2^2$ with which we started. So this is, in fact, the general second-order case. For completeness we note that after this reduction

$$\begin{aligned}\Theta &= \begin{pmatrix} \mathrm{d}\alpha/\mathrm{d}t + \beta & d\beta/\mathrm{d}t - \alpha\omega_2^2 \\ \alpha & \beta \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{d}\theta_1/\mathrm{d}t & \mathrm{d}\theta_2/\mathrm{d}t \\ \theta_1 & \theta_2 \end{pmatrix} \begin{pmatrix} d\tilde{\theta}_1/\mathrm{d}t & d\tilde{\theta}_2/\mathrm{d}t \\ \tilde{\theta}_1 & \tilde{\theta}_2 \end{pmatrix}^{-1}\end{aligned}$$

$\theta_1, \theta_2, \tilde{\theta}_1$ and $\tilde{\theta}_2$ being as before, a formula from which we easily recover the expressions (12) and (13).

There will be similar connections between families of third- and higher-order linear equations and linear systems with special classes of nonlinear equation analogous to these Ermakov–Pinney systems. They will be of interest and presumably reducible to simple quadrature in the same way.

References

[1] Athorne C, Rogers C, Ramgulam U and Osbaldestin A O 1990 *Phys. Lett.* A **143** 207–12
[2] Athorne C 1991 *J. Phys. A: Math. Gen.* **24** 945–61
[3] Athorne C 1992 *J. Diff. Eqns* **100** 82–94
[4] Athorne C 1991 *Phys. Lett.* A **159** 375–8
[5] Haas F and Goedert J J 1999 *J. Phys. A: Math. Gen.* **32** 2835–44
[6] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons (Series in Nonlinear Dynamics)* (Berlin: Springer)
[7] van der Put M. 1999 *J. Symbolic Comput.* **28** 441–72
[8] Ray J R and Reid J L 1980 *Lett. Math. Phys.* **4** 235–40